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# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

## TECHNICAL MEMORANDUM 1346

# NONLINEAR THEORY OF A HOT-WIRE ANEMOMETER

By R. Betchov

Translation of "Théorie non-linéaire de l'anémomètre à fil chaud."
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We study here the properties of a hot-wire anemometer under the supposition that the heat transfer from the wire to the air depends, first, on the difference in temperature and, second, on the square of that difference. This latter hypothesis is confirmed by experience, and the consequences might be of great importance, that effect of nonlinearity is stronger than the effect of thermal conduction.

## I. THE NONLINEAR LAW OF KING

The heat quantity $Q$ removed per second by an air stream $V$ from a wire of the diameter $d$ and unit length is given by King in the form

$$Q = (a + b\sqrt{Vd})T \tag{1}$$

where $T$ denotes the temperature difference between wire and air. King's calculation, approximately confirmed by experience, yields

$$a = \varkappa' \qquad b = \sqrt{2\pi\varkappa'\delta'c'} \tag{2}$$

with $\varkappa'$ = thermal conductivity of the air, $\delta'$ = density of the air, and $c'$ = specific heat of the air for constant volume.

These quantities may vary with $T$, and experience shows that $a$ increases while $b$ remains practically constant. Intuitively, one may interpret this effect by saying that the air in contact with the wire is heated which increases its conductivity. In compensation, its density decreases because the pressure varies only very slightly. Obviously, the effects on $\varkappa'$ and $\delta'$ compensate one another, and only $a$ varies.

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King states in his original report (ref. 1) that $a$ increases by 0.114 percent per degree; he also describes there an effect of the diameter on that term $a$ which we shall not discuss here. Thus it is advisable to write

$$Q = \left\{a(1 + \gamma T) + b\sqrt{Vd}\right\} T \tag{3}$$

where $\gamma$ takes the nonlinearity into account.

One must not forget the hypotheses on which King bases his calculation: he contends that the air flow is without viscosity and that the heat flow in the immediate proximity of the wire is constant. He uses the specific heat at constant volume although the pressure is certainly more constant than the density. For that reason, we consider equation (3) as an empirical relation, valid for the wire unit length, and would wish to see King's problem made the subject of a more thorough investigation.

Here we intend to study the effect of the term $\gamma$ on the properties of the hot wire; we simplify the notation by introducing $P$ so that

$$P = \frac{b}{a}\sqrt{Vd} \tag{4}$$

$$Q = a(1 + P + \gamma T)T \tag{5}$$

## II. GENERAL EQUATION OF THE HOT WIRE

We shall use the following symbols:

$S$ resistance of the wire, per unit length, at operating temperature

$S_o$ resistance of the wire, per unit length, at ambient temperature

$I$ intensity of the electric current heating the wire

$\alpha$ coefficient of the variation of $S$ according to the temperature

$m$ weight of the wire, per unit length

c specific heat of the wire, in joule/gram and degree

$\varkappa$ coefficient of the thermal conductivity of the wire in watt/cm and degree

$\sigma$ wire cross-sectional area

$l$ semilength of the wire

t time

x coordinate of position, varying from $l$ to $-l$

We put

$$A = \frac{a}{\alpha S_o}(1 + P) = \frac{a + b\sqrt{Vd}}{\alpha S_o} \tag{6}$$

The equation of the hot wire must express the equilibrium between the heat supplied per second, the heat removed by the air stream, the heat required to raise the temperature of the wire, and the heat transmitted by conduction. One obtains

$$SI^2 = A(S - S_o) + \frac{a\gamma}{\alpha^2 S_o^2}(S - S_o)^2 + \frac{mc}{\alpha S_o}\frac{\partial S}{\partial t} - \frac{\varkappa\sigma}{\alpha S_o}\frac{\partial^2 S}{\partial x^2} \tag{7}$$

For the steady-state case, and introducing the parameters

$$\left.\begin{aligned} y = \frac{x}{l^*} \qquad l^* = \sqrt{\frac{\varkappa\sigma}{\alpha S_o(A - I^2)}} \qquad Z = \frac{A - I^2}{S_o I^2}(S - S_o) \\ G = \frac{2}{3}\frac{a\gamma}{\alpha^2 S_o}\frac{I^2/A^2}{(1 - I^2/A)^2} = \frac{2}{3}\frac{\gamma}{\alpha}\frac{1}{1 + P}\frac{I^2/A}{(1 - I^2/A)^2} \end{aligned}\right\} \tag{8}$$

one obtains the equation (7) in the form

$$Z + \frac{3}{2}GZ^2 - \frac{\partial^2 Z}{\partial y^2} = 1 \tag{9}$$

## III. EXACT INTEGRATION OF THE STATIC CASE

Multiplying equation (9) by $\partial Z/\partial y$ and integrating, one obtains, with a constant

$$\frac{\partial Z}{\partial y} = \sqrt{GZ^3 + Z^2 - 2Z + \text{const}} \tag{10}$$

One notes that $\partial Z/\partial y$ is zero for a negative value of $Z$ and may be zero for two positive values of $Z$. At the ends of the wire, one has $S = S_0$ and $Z = 0$; at the center, $Z$ must be positive and $\partial Z/\partial y$ zero. The range of interest for us lies, therefore, between $Z = 0$ and the first positive root which gives $\frac{\partial Z}{\partial y} = 0$ which we shall denote by $Z = B$. We put

$$Z(y) = B - X^2(y) \tag{11}$$

By virtue of the relation

$$GB^3 + B^2 - 2B + \text{const} = 0 \tag{12}$$

one obtains

$$4(\partial X/\partial y)^2 = -GX^4 + EX^2 + D \tag{13}$$

with

$$E = 1 + 3GB \qquad D = 2(1 - B) - 3GB^2 \tag{14}$$

Following, we shall consider $B$ as a new integration constant indicating the temperature in the middle of the wire. At the center of the wire, one has $X = 0$ and

$$4(\partial X/\partial y)^2 = D \tag{15}$$

which shows that $D$ is positive and generally small. At the ends of the wire, $Z = 0$ and $X = \pm\sqrt{B}$.

The roots of equation (13) are

$$X^2 = \frac{-E \pm\sqrt{E^2 + 4GD}}{-2G} \tag{16}$$

Introducing the parameter $\beta$ so that

$$\sinh \beta = \text{sh } \beta = \frac{2\sqrt{GD}}{E} \tag{17}$$

one can write equation (13) in the form

$$4(\partial X/\partial y)^2 = G\left[\frac{E}{2G}(\text{ch } \beta + 1) - X^2\right]\left[\frac{E}{2G}(\text{ch } \beta - 1) + X^2\right] \tag{18}$$

We define the angle $\varphi$, function of $y$, so that

$$X = \sqrt{\frac{D}{E \text{ ch } \beta}}\frac{\sin \varphi}{\sqrt{1 - k^2\sin^2\varphi}} \tag{19}$$

with

$$k^2 = \frac{\text{ch } \beta + 1}{2 \text{ ch } \beta} \tag{20}$$

Equation (18) then becomes

$$\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = \frac{1}{2}\sqrt{E \text{ ch } \beta}\, dy \tag{21}$$

and we obtain the elliptic integral of the first kind

$$U(k;\varphi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = \frac{1}{2}\sqrt{E \text{ ch } \beta}\, y \tag{22}$$

The variable $y$ varies from $-\xi$ to $\xi$, with

$$\xi = l/l^* \tag{23}$$

and we have, for $y = \xi$ and $X^2 = B$

$$\frac{\xi}{2}\sqrt{E \text{ ch } \beta} = \int_0^{\varphi_{max}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \tag{24}$$

with

$$\sqrt{B} = \sqrt{\frac{D}{E \text{ ch } \beta}} \frac{\sin \varphi_{max}}{\sqrt{1 - k^2 \sin^2\varphi_{max}}} \tag{25}$$

This last equation may be written

$$\frac{1}{\sin^2\varphi_{max}} = k^2 + \frac{D}{EB \text{ ch } \beta} \tag{26}$$

From equation (20) one may deduce

$$\frac{1}{k^2} = 1 + \tanh^2(\beta/2) \tag{27}$$

With $\beta$ ranging from $0$ to $+\infty$, $k^2$ varies from 1 to 0.5.

With G and B known, one may calculate successively D, E, $\beta$, k, and $\varphi_{max}$. A table of $U(k,\varphi)$ then permits us to calculate $\xi$. Figure 1 gives the results obtained by this procedure and allows - starting out from a prescribed wire and with $\gamma$ known - to determine B from G and $\xi$.

The temperature distribution over the wire is given by Z as a function of y, or by X as a function of $\varphi$ and $\varphi$ a function of y. The relation $\varphi(y)$ is given by the quotient of equations (22) and (24), namely

$$\frac{y}{\xi} = \frac{\int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}}{\int_0^{\varphi_{max}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}} \tag{28}$$

From equations (19) and (25), one obtains a relation between $\varphi$ and X, namely

$$\frac{X^2}{B} = \frac{\sin^2\varphi}{\sin^2\varphi_{max}} \frac{1 - k^2 \sin^2\varphi_{max}}{1 - k^2 \sin^2\varphi} \tag{29}$$

The total resistance R of the wire is given by

$$R = \int_{-l}^{+l} S\,dx = \frac{2S_o I^2}{A - I^2} l^* \int_0^{\xi} Z\,dy + 2S_o l \tag{30}$$

We introduce the cold resistance $R_o$ and the function X

$$R - R_o = \frac{R_o I^2}{A - I^2}\left(B - \frac{1}{\xi}\int_0^{\xi} X^2\,dy\right) \tag{31}$$

We replace X according to equation (19) and dy according to equation (21), namely

$$R - R_o = \frac{R_o I^2}{A - I^2}\left(B - \frac{2D}{\xi(E\ \mathrm{ch}\ \beta)^{3/2}}\int_0^{\varphi_{max}} \frac{\sin^2\varphi}{(1 - k^2\sin^2\varphi)^{3/2}}\, d\varphi\right) \tag{32}$$

The integral is equal to $2\partial U/\partial k^2$ and we obtain

$$R - R_o = \frac{R_o I^2}{A - I^2}\left(B - \frac{2D}{E\ \mathrm{ch}\ \beta}\,\frac{\partial U/\partial k^2}{U}\right) \tag{33}$$

The values of $\partial U/\partial k^2$ can be deduced from a good table of $U(k,\varphi)$ with sufficient approximation.

If the wire were perfect, the expression in parentheses in equation (33) would have to be replaced by unity; therefore, we shall introduce the quantity M so that

$$M = 1 - B + \frac{2D}{E\ \mathrm{ch}\ \beta}\,\frac{\partial U/\partial k^2}{U} \tag{34}$$

The formula (33) then gives us

$$R = R_o\,\frac{1 - MI^2/A}{1 - I^2/A} \tag{35}$$

and the important relation

$$\frac{RI^2}{R - R_o} = A\left\{1 + \frac{M}{1 - M}\left(1 - I^2/A\right)\right\} \tag{36}$$

This last equation permits easy determination of the wire characteristics because R, $R_o$, and I can be measured accurately and because the curves obtained as functions of $R/R_o$, for instance, indicate directly the effects of conduction and of nonlinearity. We calculated the values of $\frac{M}{1 - M}$ as a function of G and $\xi$; figure 2 shows our results.

A good approximation is given by

$$\frac{M}{1 - M} = \frac{1}{B_o} - 1 + \frac{1}{\xi - 1} \tag{37}$$

where $B_o$ corresponds to B, in the case $\xi = \infty$

$$B_o = \frac{+\sqrt{1 + 6G} - 1}{3G} \tag{38}$$

## IV. A FEW USEFUL APPROXIMATIONS

In performing the calculations necessary for the plotting of figure 1, we have noted that one may assign to k the value unity without introducing large errors.

This implies $\beta = 0$ and equation (19) then gives

$$X = \sqrt{\frac{D}{E}} \tan \varphi \tag{39}$$

The integral (22) becomes

$$U = \int_0^{\varphi} \frac{d\varphi}{\cos \varphi} = \frac{1}{2} L \frac{1 + \sin \varphi}{1 - \sin \varphi} = \frac{1}{2} \sqrt{E}y \tag{40}$$

whence, one deduces

$$\mathrm{ch}(2U) = \frac{1 + \sin^2\varphi}{1 - \sin^2\varphi} = \mathrm{ch}(\sqrt{E}y) \tag{41}$$

$$X^2 = \frac{D}{2E}(\mathrm{ch}\,\sqrt{E}y - 1) \tag{42}$$

At the limits, one has

$$B = \frac{D}{2E}(\mathrm{ch}\,\sqrt{E}\xi - 1) \tag{43}$$

which gives

$$Z = \frac{B}{1 - 1/\mathrm{ch}\,\sqrt{E}\xi}\left(1 - \frac{\mathrm{ch}\,\sqrt{E}y}{\mathrm{ch}\,\sqrt{E}\xi}\right) \tag{44}$$

In order to calculate $M$, one must put

$$\lim_{k \longrightarrow 1} \partial U/\partial k^2 = \frac{1}{2}\int_0^{\varphi} \frac{\sin^2\varphi}{\cos^3\varphi}\, d\varphi = \frac{1}{4}\left(\frac{\sin\,\varphi}{\cos^2\varphi} - U(\varphi)\right) \tag{45}$$

which gives

$$M = 1 - \frac{B}{1 - 1/\mathrm{ch}\,\sqrt{E}\xi}\left(1 - \frac{\mathrm{Th}\,\sqrt{E}\xi}{\sqrt{E}\xi}\right) \tag{46}$$

One can see that, due to the nonlinearity, $\xi$ is replaced by $\sqrt{E}\xi$, and the central temperature is lowered.

If one takes equation (44) as solution of equation (9), one sees that the equation is satisfied for a term of approximately $(\mathrm{ch}\,\sqrt{E}y/\mathrm{ch}\,\sqrt{E}\xi)^2$, and that $B$ is given approximately by

$$B = \left(\frac{\sqrt{1 + 6G} - 1}{3G}\right)(1 - 1/\mathrm{ch}\,\sqrt{E}\xi) \tag{47}$$

with

$$E \simeq \sqrt{1 + 6G} \tag{48}$$

We shall take an example that represents an extreme case. We choose a platinum wire with 10 percent of iridium, a diameter of 7 microns, and a length of $2l = 1.14$ mm. Exposed to an air stream of 5 m/sec and heated with 75 mA, it gives us

$$P = 2.9 \qquad A = 1.2 \times 10^{-2} \qquad I^2/A = 0.47$$

$$l^* = 0.15 \text{ mm} \qquad \xi = 3.75$$

Assuming $\gamma = 1.2 \times 10^{-3}$ and with the aid of figures 1 and 2, we determined

$$G = 0.25 \qquad B = 0.76 \qquad E = 1.56 \qquad M = 0.4$$

The other parameters have the following calculated values:

$$k = 0.99756 \qquad \varphi_{max} = 79.5^{\circ}$$

$$\text{sh } \beta = 0.14 \qquad D = 0.0477 \qquad U = 2.34$$

In figure 3, we show the profile of the temperatures calculated exactly, the profile calculated with the approximation $k = 1$, and the profile calculated with $\gamma = 0$. It can be seen that the nonlinearity offers a more uniform temperature distribution, and that the approximation is sufficient.

By means of equation (35) one calculates $\frac{R}{R_o} = 1.5$, whereas the calculation with $\gamma = 0$ would give the result 1.9. The mean temperature giving $\frac{R}{R_o} = 1.5$ would be 380°.

As to the term $RI^2/R - R_o$, it changes from the value 1.24 A when the current is very weak to the value 1.35 A when $I$ attains 75 mA.

It varies therefore by about 10 percent between $\frac{R}{R_o} = 1$ and $\frac{R}{R_o} = 1.5$ which is of the same order as the variations observed.

Thus this magnitude is constant at 10 percent and King's law is verified; however, we shall see later on that the thermal inertia is very different from the expected value.

## V. THE DYNAMIC EQUATION OF THE HOT WIRE

In order to calculate the variation of the total resistance of the wire when the current or the air stream fluctuate, one must go back to the equation (7). Replacing in this equation I, S, and V (contained in the term A) by $I + ie^{j\omega t}$, $S + se^{j\omega t}$, and $V + ve^{j\omega t}$, one obtains after suppressing the terms of the order zero, two, and more as well as the factor $e^{j\omega t}$

$$2SIi + I^2 s = \frac{aP}{2\alpha S_o}\frac{v}{V}(S - S_o) + As + \frac{2a\gamma}{\alpha^2 S_o{}^2}(S - S_o)s + \frac{mc}{\alpha S_o} j\omega s - \frac{\chi\sigma}{\alpha S_o}\frac{\partial^2 s}{\partial x^2} \tag{49}$$

We introduce

$$z = s\,\frac{A - I^2}{S_o I^2} \tag{50}$$

and, identical to the formula (9) of our Mededeling No. 55 (ref. 5)

$$\omega^* = \frac{\alpha S_o}{mc}(A - I^2) \tag{51}$$

One then obtains, after introduction of Z

$$2\,\frac{i}{I} + 2\,\frac{i}{I}\,\frac{I^2/A}{1 - I^2/A}\,Z - \frac{1}{2}\,\frac{P}{1 + P}\,\frac{Z}{1 - I^2/A}\,\frac{v}{V} = z(1 + 3GZ + j\omega/\omega^*) - \frac{\partial^2 z}{\partial y^2} \tag{52}$$

The function $Z(y)$ appears twice in this differential equation, and we must utilize the approximation $k = 1$ in order to avoid great complications. Writing the expression $Z$ according to equation (44), one obtains

$$-\frac{\partial^2 z}{\partial y^2} + z\left(1 + \frac{3GB}{1 - 1/\operatorname{ch}\sqrt{E}\xi} + j\omega/\omega^* - \frac{3GB}{\operatorname{ch}\sqrt{E}\xi - 1}\operatorname{ch}\sqrt{E}y\right) = C_1 + C_2\frac{\operatorname{ch}\sqrt{E}y}{\operatorname{ch}\sqrt{E}\xi} \tag{53}$$

with

$$C_1 = 2\frac{i}{I}\left(1 + \frac{I^2/A}{1 - I^2/A}\frac{B}{1 - 1/\operatorname{ch}\sqrt{E}\xi}\right) - \frac{1}{2}\frac{v}{V}\left(\frac{P}{1 + P}\frac{1}{1 - I^2/A}\frac{B}{1 - 1/\operatorname{ch}\sqrt{E}\xi}\right) \tag{54}$$

$$C_2 = -2\frac{i}{I}\frac{I^2/A}{1 - I^2/A}\frac{B}{1 - 1/\operatorname{ch}\sqrt{E}\xi} + \frac{1}{2}\frac{v}{V}\left(\frac{P}{1 + P}\frac{1}{1 - I^2/A}\frac{B}{1 - 1/\operatorname{ch}\sqrt{E}\xi}\right) \tag{55}$$

The solution without the second member of equation (53) is a Mathieu function taken for a purely imaginary value of the argument, but if $\sqrt{E}\xi$ is large enough, for instance, more than 4, one may neglect the term with $\operatorname{ch}\sqrt{E}y$ of the member on the left side of equation (53). Actually it is important only when $y$ is close to $\xi$; however, we shall take as a limiting condition $z(\xi) = 0$, and the product $z \operatorname{ch}\sqrt{E}y$ will never be important. We shall also neglect the term $1/\operatorname{ch}\sqrt{E}\xi$ in the denominator of one of the terms on the left side of equation (53) which amounts to taking $Z = B$ in the factor term of $z$.

Taking the definition of $E$ into account, one then has

$$-\frac{\partial^2 z}{\partial y^2} + z(E + j\omega/\omega^*) = C_1 + C_2\frac{\operatorname{ch}\sqrt{E}y}{\operatorname{ch}\sqrt{E}\xi} \tag{56}$$

the solution of which - null for $y = \pm\xi$ - is

$$z = \frac{C_1}{Ep^2}\left(1 - \frac{\operatorname{ch} p\sqrt{E}y}{\operatorname{ch} p\sqrt{E}\xi}\right) + \frac{C_2}{j\omega/\omega^*}\left(\frac{\operatorname{ch}\sqrt{E}y}{\operatorname{ch}\sqrt{E}\xi} - \frac{\operatorname{ch} p\sqrt{E}y}{\operatorname{ch} p\sqrt{E}\xi}\right) \tag{57}$$

with

$$p = \sqrt{1 + j\omega/E\omega^*} \tag{58}$$

It can be seen that due to the nonlinearity the characteristic frequency $\omega^*$ is replaced by a new frequency which is higher by a factor E. (Compare with equation (8) of ref. 5.) The amplitudes according to $C_1$ and $C_2$ also are modified by the nonlinearity.

The variation r of the total resistance R will be

$$r = \int_{-l}^{+l} s\,dx = \frac{R_o I^2}{A - I^2}\frac{1}{\xi}\int_0^{\xi} z\,dy \tag{59}$$

The integration gives

$$r = \frac{R_o I^2}{A - I^2}\left\{\frac{C_1}{Ep^2}\left(1 - \frac{\tanh p\sqrt{E}\xi}{p\sqrt{E}\xi}\right) + \frac{C_2}{j\omega/\omega^*}\frac{1}{\sqrt{E}\xi}\left(\tanh\sqrt{E}\xi - \frac{\tanh p\sqrt{E}\xi}{p}\right)\right\} \tag{60}$$

We assumed above that $\sqrt{E}\xi$ is sufficiently large; also, we may assign to the hyperbolic tangents the value unity (the presence of a complex argument is here not of importance).

One then obtains

$$r = \frac{R_o I^2}{A - I^2}\left\{\frac{C_1}{Ep^2}(1 - 1/p\sqrt{E}\xi) + \frac{C_2}{j\omega/\omega^*}\frac{1}{\sqrt{E}\xi}\left(\frac{p - 1}{p}\right)\right\} \tag{61}$$

## VI. RESPONSE TO A FLUCTUATION OF THE CURRENT

If one assumes $v = 0$ in equations (54) and (55), one may transform equation (61), suppressing the terms containing $1/\mathrm{ch}\sqrt{E}\xi$

$$rI = \frac{2iR_o I^2/A}{(1 - I^2/A)^2}\left\{\frac{(1 - I^2/A(1 - B))(1 - 1/p\sqrt{E}\xi)}{Ep^2} - \frac{BI^2/A}{\sqrt{E}\xi}\frac{\omega^*}{j\omega}\left(\frac{p - 1}{p}\right)\right\} \tag{62}$$

This formula gives the alternating electromotive force produced at the boundaries of the hot wire by the modulation current $i$ in addition to the normal electromotive force $Ri$. Although this formula seems to be complicated, it can be adapted to the needs of practice. If the frequency $\omega/2\pi$ tends toward infinity, equation (62) becomes

$$rI = \frac{2iR_oI^2/A}{(1 - I^2/A)^2} \frac{(1 - I^2/A(1 - B) - I^2/AB/\sqrt{E\xi})}{j\omega/\omega^*} \tag{63}$$

If one takes into account that equation (35) gives $R$, that equation (46) gives $M$, and that equation (51) gives $\omega^*$, one finds

$$rI = \frac{-j}{\omega/2\pi} iCRI^2 \qquad \text{with} \qquad C = \frac{\alpha S_o}{\pi mc} \qquad (64) \quad (65)$$

The constant $C$ corresponds to that of our former publications, and equation (64) shows that the electromotive force $rI$ taken at high frequency permits to measure $C$ without being impeded by either conduction or nonlinearity. The method described in reference 4 is therefore indicated rather than the one consisting of measuring the phase displacements, with $\omega$ being of the order of $\omega^*$.

One may immediately verify this point by assuming $\omega$ in the equation (49) as very large, thus making the effect of the terms of conduction and of nonlinearity negligible.

When $\omega$ tends toward zero, the electromotive force becomes

$$rI = \frac{2iR_oI^2/A}{(1 - I^2/A)^2} \frac{1}{E} \left\{1 - \frac{I^2}{A} M - \frac{1}{\sqrt{E\xi}}\left(1 - I^2/A\left(1 - \frac{1}{2} B\right)\right)\right\} \tag{66}$$

Thus the complex function $rI$ according to equation (62) will change from equation (64) into equation (66) when $\omega$ varies from 0 to a large value. The complex trace of this function gives practically a semicircle which permits to put approximately

$$rI = \frac{rI(\omega = 0)}{1 + j\omega/\omega^{**}} \tag{67}$$

where $\omega^{**}$ denotes the effective characteristic frequency. In figure 4, we plotted the semicircle and indicated a few values of $\omega/\omega^{**}$. From equation (62), and in the case of the preceding example, we calculated the electromotive forces rI for a few values of $\omega/E\omega^*$. One can see that the two functions blend, at low frequencies, if one assumes $\omega^{**} = 1.1E\omega^* = 1.72\omega^*$. At high frequency, equations (64) and (67) will be equal which permits calculation of a satisfactory value of $\omega^{**}$ when the effect of thermal inertia is important. One obtains

$$\omega^{**} = \frac{E\omega^*}{1 - 1/\sqrt{E}\xi \left\{\frac{1 - I^2/A\left(1 - \frac{1}{2} B\right)}{1 - I^2/AM}\right\}} \tag{68}$$

In the case treated one finds $\omega^{**} = 1.23E\omega^*$, that is, $\omega^{**} = 1.92\omega^*$: the effective characteristic frequency is almost twice the expected value; therefore, the approximation (67) gives a correct plot of the function, but the phases according to equation (68) will be only within a 10-percent accuracy.

The denominator of equation (68) depends chiefly on $\xi$, and it increases the characteristic frequency. Instead of compensating each other as in the static case, the two effects reduce the thermal inertia.

Intuitively, one may say that the conduction shortens the hot part of the wire and thus reduces the heat required for modifying the central temperature; the nonlinearity depends on the presence of hot air around the wire, and the thermal inertia of the air is negligible which improves the spherical response of the anemometer.

When the wire is dusty, the quantity of immobile air is greater, and the experience shows that the term a in equation (3) is increased while b remains unchanged. One must therefore expect a dynamic action of the dust of the wire to the extent that E is modified. The dynamic effect may be more important than the static effect.

The wire in the quoted example demonstrates that, with a term $RI^2/R - R_o$ constant at 10 percent, the characteristic frequency may be almost twice the normally foreseen value.

## VII. RESPONSE TO A FLUCTUATION OF THE AIR STREAM

Assuming $i = 0$ in the formulas (54) and (55), and maintaining $v$, one may transform equation (61) into

$$\left.\begin{aligned} rI = -\frac{1}{2}\frac{v}{V}\frac{P}{1+P}\frac{R_o I^3/A}{(1-I^2/A)^2}\frac{B}{1-1/\operatorname{ch}\sqrt{E\xi}} \\ \left\{\frac{1-1/p\sqrt{E\xi}}{Ep^2}-\frac{\omega^*}{j\omega}\frac{p-1}{p}\frac{1}{\sqrt{E\xi}}\right\}\end{aligned}\right\} \tag{69}$$

With $\omega$ tending toward zero, one has

$$rI = -\frac{1}{2}\frac{v}{V}\frac{P}{1+P}\frac{R_o I^3/A}{(1-I^2/A)^2}\frac{B}{1-1/\operatorname{ch}\sqrt{E\xi}}\left\{\frac{1-\frac{3}{2}\frac{1}{\sqrt{E\xi}}}{E}\right\} \tag{70}$$

If $\omega$ tends toward infinity, one has

$$rI = -\frac{1}{2}\frac{v}{V}\frac{P}{1+P}\frac{R_o I^3/A}{(1-I^2/A)^2}\frac{B}{1-1/\operatorname{ch}\sqrt{E\xi}}\left\{\frac{1-\frac{1}{\sqrt{E\xi}}}{j\omega/\omega^*}\right\} \tag{71}$$

and one may approximately replace equation (69) by the semicircle

$$rI = \frac{rI(\omega = 0)}{1 + j\omega/\omega^{**}} \tag{72}$$

with

$$\omega^{**} = E\omega^* \frac{1 - 1/\sqrt{E\xi}}{1 - \frac{3}{2}1/\sqrt{E\xi}} \tag{73}$$

which, in the case of the example treated above, gives $\omega^{**} = 1.81\omega^*$.

Thus there is, on principle, no equality between the dynamic reaction to a variation $i$ and to a variation $v$.

This difficulty arises due to the term $\sqrt{\overline{E}}\xi$, namely to the conduction; the nonlinearity tends toward diminishing its importance (factor $\sqrt{\overline{E}}$).

We hope to publish some empirical results, and the calculation of the differences indicated by different authors, in the near future.

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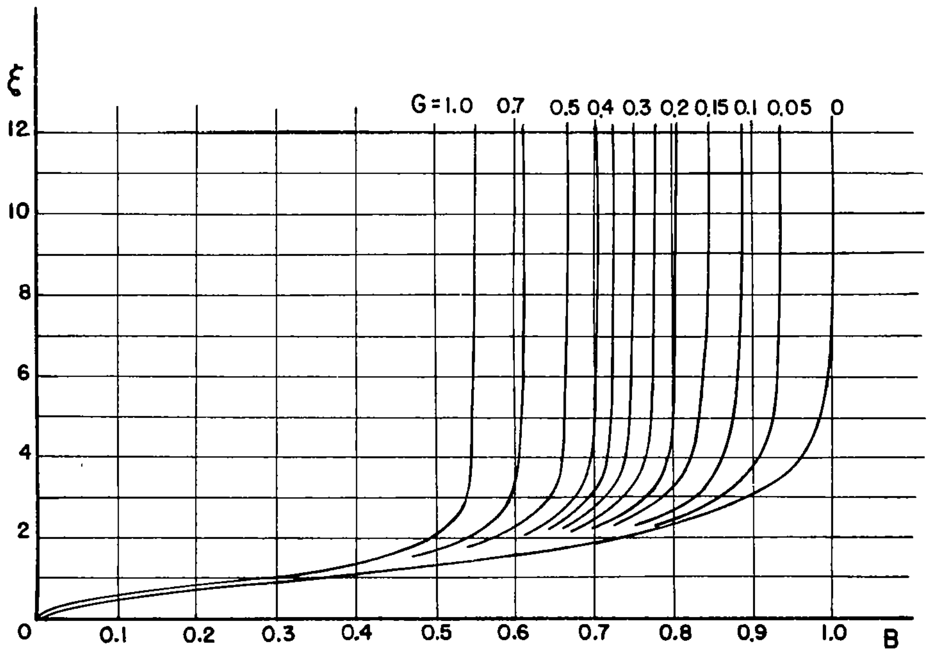


Figure 1.- Relation between the magnitudes B, G, and $\xi$.

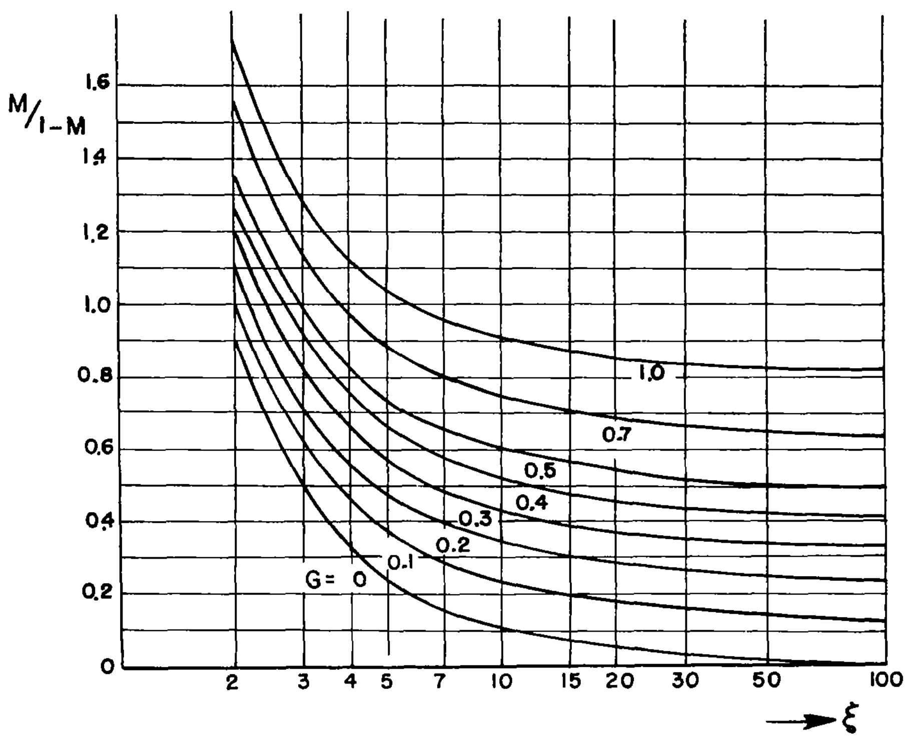


Figure 2.- Values of M/1 - M according to G and ξ.

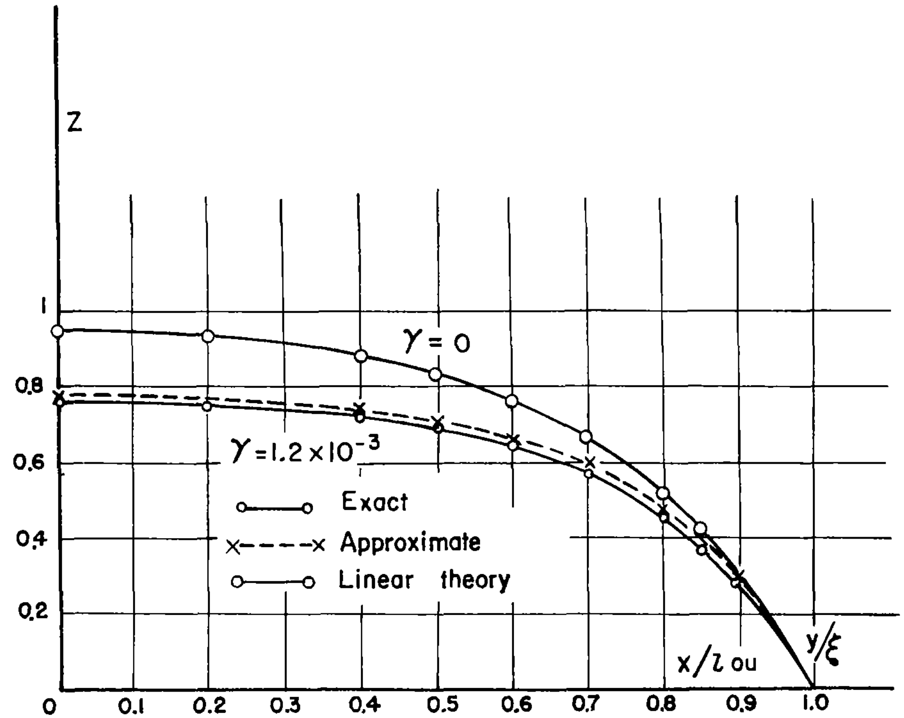


Figure 3.- Temperature distribution (with a wire of 7 microns) according to linear theory and according to our formulas.

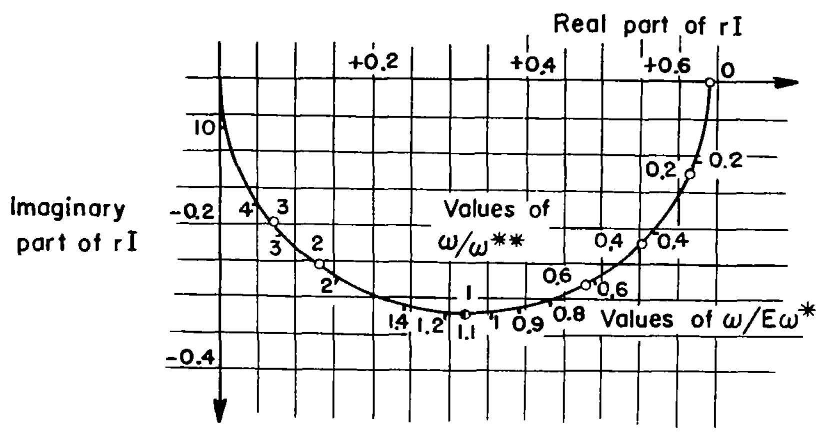


Figure 4.- Study of the tension rI. The semicircle corresponds to formula (67); the points give rI, according to (62), for several values of $\omega/E\omega^*$.